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Racah coefficients of quantum group $U_q(n)$

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Abstract. Racah coefficients of $U_q(n)$ are derived from subduction coefficients of Hecke algebras $H_f(q)$ by using the Schur–Weyl duality relation between $U_q(n)$ and $H_f(q)$. The Racah coefficients of $U_q(n)$ for the resulting irreps $[n_1, n_2, \ldots, n_k, \dot{0}]$ with $\Sigma_{i=1}^k n_i \leqslant 5$, and some of them for the resulting irreps with multiplicity and $\Sigma_{i=1}^k n_i \leqslant 6$ are tabulated.

1. Introduction

Quantized universal enveloping algebras [1–3], often referred to as quantum groups or quantum algebras, arise as underlying symmetries of integrable models [4] and conformal field theory [5]. The Racah–Wigner algebra plays a fundamental role in the representation theory of Lie algebras, and its importance to the representation theory of quantum group is expected in connection with IRF models in statistical mechanics [6] and the determination of new knot invariants [7]. The Racah coefficients are also related to the braiding and fusing matrices in the RCFT [5, 8].

The Racah coefficients in the 6*j* symbol form of $SU_q(2)$ were first presented by Kirillov and Reshetikhin [9], and were then discussed by many authors using different methods [10–15]. Recently, some properties of the Racah coefficients for quantum groups were given in [16], in which 6*j* symbols for $SU_q(2)$ are used to illustrate the building-up method for calculating the coupling and recoupling coefficients.

In this paper, based on the well known Schur–Weyl duality relation between $U_q(n)$ and $H_f(q)$, a method for the evaluation of $U_q(n)$ Racah coefficients from subduction coefficients (SDCs) of the Hecke algebra $H_f(q)$ will be outlined. The advantage of this method is that the calculation of the $U_q(n)$ Racah coefficients is rank-independent. The Racah coefficients of $U_q(n)$ for the resulting irreps $[n_1, n_2, \ldots, n_k, \dot{0}]$ with $\Sigma_{i=1}^k n_i \leqslant 5$, and some of them for the resulting irreps with multiplicity and $\Sigma_{i=1}^k n_i \leqslant 6$, will be tabulated.

2. Evaluation of the $U_q(n)$ Racah coefficients

The $U_q(n)$ Racah coefficients are simply a generalization of the $SU_q(2)$ Racah coefficients, which are the elements of a unitary matrix between bases with two different coupling orders of three irreps v_1, v_2 and v_3 of $U_q(n)$,

$$|(v_1 v_2)v_{12}, v_3; vw\rangle_q^{t_{12}t} = \sum_{\substack{v_{23}t_{23}\\ t'}} U_q(v_1 v_2 v v_3; v_{12} v_{23})_{t_{23}t'}^{t_{12}t} |v_1(v_2 v_3)v_{23}; vw\rangle_q^{t_{23}t'} \tag{2.1}$$

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where four multiplicity labels appeared,

$$\begin{aligned} &t_{12}=1,2,\ldots,\{v_1, v_2v_{12}\} \qquad && t_{23}=1,2,\ldots,\{v_2v_3v_{23}\} \\ &t=1,2,\ldots,\{v_{12}v_3v\} \qquad && t'=1,2,\ldots,\{v_1v_{23}v\}. \end{aligned} \tag{2.2}$$

The $U_q(n)$ Racah coefficient can be expressed in terms of the $U_q(n)$ CG coefficients,

$$U_q(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} = \sum_{\text{fix } w} C^{v_{12}t_{12}w_{12}}_{v_1w_1,v_2w_2}(q)\, C^{vtw}_{v_{12}w_{12},v_3w_3}(q)\, C^{v_{23}t_{23}w_{23}}_{v_2w_2,v_3w_3}(q)\, C^{vt'w}_{v_1w_1,v_{23}w_{23}}(q) \tag{2.3a}$$

where the summation is carried out for all possible component indices under the condition that w is fixed. Some $U_q(n)$ CG coefficients have already been derived in [23]. The Racah coefficients involve four triples of partitions:

$$\Delta(v_1v_2v_{12}) \qquad \Delta(v_2v_3v_{23}) \qquad \Delta(v_{12}v_3v) \qquad \Delta(v_1v_{23}v). \tag{2.3b}$$

A Racah coefficient is zero whenever one of the triple of partitions does not satisfy the generalized triangular relation decided by the Littlewood rule. The Racah coefficients satisfy the unitarity conditions

$$\begin{aligned} &\sum_{\substack{v_{23}t_{23}\\ t'}} U_q(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'}\, U_q(v_1v_2vv_3;\, \bar{v}_{12}v_{23})^{\rho_{12}\rho}_{t_{23}t'} = \delta_{t_{12}\rho_{12}}\delta_{t\rho}\delta_{v_{12}\bar{v}_{12}} \\ &\sum_{\substack{t_{12}v_{12}\\ t}} U_q(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'}\, U_q(v_1v_2vv_3;\, v_{12}\bar{v}_{23})^{t_{12}t}_{\rho_{23}\rho'} = \delta_{t_{23}\rho_{23}}\delta_{t'\rho'}\delta_{v_{23}\bar{v}_{23}}. \end{aligned} \tag{2.4}$$

Formula (2.3) is not suitable for practical computation of the $U_q(n)$ Racah coefficients. However, there is an important relation, the so-called Schur–Weyl duality relation between the quantum group $U_q(n)$ and the Hecke algebra $H_f(q)$, which was first observed by Jimbo [3], and was also studied by many others [17–20]. That is, the images of the $U_q(n)$ and $H_f(q)$ generators vary continuously with q; and the algebra $H_f(q)$ and $U_q(n)$ are commutants of one another in the tensor space $(V^n)^{\otimes f}$. Hence, for generic q the structure of $H_f(q)$ can be determined from the information about $H_f(1)=CS_f$, which is the group algebra of the symmetric group S_f; and a standard basis of $H_f(q)$ for an irrep $[v]$ is also a special Gel'fand basis of $U_q(n)$ for the same irrep. Thus, from the analytical continuation for q, we can conclude that the Schur–Weyl duality between S_f and $U(n)$ applies to $H_f(q)$ and $U_q(n)$ as well except that q is a root of unity. In the following, we always assume that q is generic.

In the $q=1$ case a formula relating the $SU(n)$ Racah coefficients to the SDCs of symmetric groups was derived in [24, 25]. After analytical continuation for q, the formula relating the $U_q(n)$ Racah coefficients to the SDCs of $H_f(q)$ can be expressed as

$$\begin{aligned} U_q(v_1v_2vv_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} &= \sum_{\substack{m_{12}m_{23}\\ m}} \left\langle \begin{matrix} v \\ m \end{matrix} \middle| v, \begin{matrix} tv_{12}v_3 \\ m_{12}m_3 \end{matrix} \right\rangle_q \left\langle \begin{matrix} v_{12} \\ m_{12} \end{matrix} \middle| v_{12}, \begin{matrix} t_{12}v_1v_2 \\ m_1m_2 \end{matrix} \right\rangle_q \\ &\quad\times \left\langle \begin{matrix} v \\ m \end{matrix} \middle| v, \begin{matrix} t'v_1v_{23} \\ m_1m_{23} \end{matrix} \right\rangle_q \left\langle \begin{matrix} v_{23} \\ m_{23} \end{matrix} \middle| v_{23}, \begin{matrix} t_{23}v_2v_3 \\ m_2m_3 \end{matrix} \right\rangle_q \end{aligned} \tag{2.5}$$

where the summation is carried out under fixed m_1, m_2 and m_3. Equation (2.5) can be used to evaluate the $U_q(n)$ Racah coefficients from the SDCs of the Hecke algebra $H_f(q)$. The SDCs of $H_f(q)$ with $f \leqslant 5$ and some of them with $f \leqslant 6$ have already been obtained by using the linear equation method (LEM) [21]. The advantage of equation (2.5) is its being rank-independent.

Due to equation (2.5) and the symmetry properties of $H_f(q)$ SDCs given by [21], the Racah coefficients have the symmetry

$$U_q(\tilde{v}_1\tilde{v}_2\tilde{v}\tilde{v}_3;\, \tilde{v}_{12}\tilde{v}_{23})^{t_{12}t}_{t_{23}t'} = \eta_1 U_q(v_1 v_2 v v_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \tag{2.6a}$$

where $\tilde{v}$ is the conjugation of v. On the other hand, due to equation (2.3a) and the symmetry properties of CG coefficients of $U_q(n)$, the Racah coefficients of $U_q(n)$ have the symmetry

$$U_q(v_3 v_2 v v_1;\, v_{23}v_{12})^{t_{23}t'}_{t_{12}t} = \eta_2 U_q(v_1 v_2 v v_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} \tag{2.6b}$$

where the phase factors η_1, η_2 are equal to

$$\eta_i = \varepsilon_i(v_1 v_2 v_{12} t_{12})\varepsilon_i(v_{12} v_3 v t)\varepsilon_i(v_2 v_3 v_{23} t_{23})\varepsilon_i(v_1 v_{23} v t') \qquad \text{for } i = 1, 2. \tag{2.6c}$$

The phase convention used here is the same as that for the symmetric groups [25]. Notice that when the multiplicity is larger than one, and the multiplicity separation is based on the *ad hoc* orthogonal procedure, the symmetries (2.6) are in general not valid. Other properties of $U_q(n)$ Racah coefficients, such as the Biedenharn–Eillot sum rule, and the Racah back-coupling rule, have already been given in [16]. It should be pointed out that the conjugation always involves a change from q to q^{-1} as noted in [16]. However, we can prove that the Racah coefficients of $U_q(n)$ are independent of q-factors. Hence

$$U_q(v_1 v_2 v v_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'} = U_{q^{-1}}(v_1 v_2 v v_3;\, v_{12}v_{23})^{t_{12}t}_{t_{23}t'}. \tag{2.7}$$

This can easily be seen from the following facts. First, the matrix elements of $U_q(n)$ generators between the canonical and non-canonical bases of $U_q(n)$ are q-factor-free, which was first pointed out by Biedenharn [26], and was verified by Jimbo [27] and Ueno *et al* [28]. Secondly, the $U_q(n)$ Racah coefficients can be expressed in terms of the SDCs of Hecke algebras, which are also the special SDCs of $U_q(m+n) \supset U_q(m) \oplus U_q(n)$; and the SDCs of $U_q(m+n) \supset U_q(m) \oplus U_q(n)$ can be derived from the matrix elements of $U_q(n)$ generators between the canonical and non-canonical bases of $U_q(n)$ by using the results of [29, 30] with q-continuation. Thus, the SDCs of Hecke algebras are also q-factor-free, which is verified by our early works [21, 22]. This conclusion can be used to check our final results.

3. Tables of $U_q(n)$ Racah coefficients

In this section, we will tabulate $U_q(n)$ Racah coefficients for the resulting irreps $[n_1, n_2, \ldots, n_k, \dot{0}]$ with $\sum_{i=1}^{k} n_i \leqslant 5$, and some of them for the resulting irreps with multiplicity and $\sum_{i=1}^{k} n_i \leqslant 6$, which are derived by using equation (2.5) and the SDCs of Hecke algebras given by [21]. Tables for the Racah coefficients (see the appendix) are arranged in the ordering of the set of irreps $(v_1 v_2 v v_3;\, v_{12}v_{23})$. The tables have the following layout:

Table X-xx. $v=\ ,v_1=\ ,v_{12}=\ .$

	$v_2\ldots$
	$v_3\ldots$
	$t(t_{12})\ldots$
v_{23}	$t'(t_{23})$
⋮	⋮

The entries are the squares of the Racah coefficients and a minus sign indicates a negative coefficient. If all vs are totally symmetric or antisymmetric, the Racah coefficient is unity and not included.

The Racah coefficients obey the symmetry (2.6*b*), where the phase is given by $\eta_2=+1$ for multiplicity-free cases and for the cases involving the triple $\Delta([21][21][321]\,t=1)$, while $\eta_2=-1$ for the cases involving the triple $\Delta([21][21][321]t=2)$. In these tables, we have defined two kinds of q-numbers, namely

$$[x]=(q^{x}-q^{-x})/(q-q^{-1}) \tag{3.1}$$

and

$$[x]'=(q^{x/2}-q^{-x/2})/(q^{1/2}-q^{-1/2}). \tag{3.2}$$

$[x]'$ appears when the Racah coefficients are non-multiplicity-free.

Only the Racah coefficients for $v_1\leqslant v_3$ are tabulated. The coefficients for $v_1>v_3$ can then be obtained from the symmetry (2.6*b*). For example, we have

$$\begin{aligned}&\mathrm{U}_q([21][2][3211][1];[32][21])_{t'=2}\\ &\quad=-\mathrm{U}_q([1][2][321][21];[21][32])^{t=2}\\ &\quad=([3]^2[5]'/2[4][2][3]'^2)^{1/2}\end{aligned} \tag{3.3}$$

from tables IV-b. Since we have used the symmetry imposition as given in [21] for $[321]\downarrow[21]\times[21]$ SDCs, our Racah coefficients do not obey the symmetry (6.6*a*) when they involve the triple $\Delta([21][21][321]t)$. For example, we have

$$\begin{aligned}&\mathrm{U}_q([1][11][321][21];[21][221])^{t=1}\\ &\quad=-\mathrm{U}_q([1][2][321][21];[21][32])^{t=2}\\ &\quad=([5]'/2[4][2])^{1/2}\end{aligned} \tag{3.4}$$

from tables IV-b.

4. Conclusions

In this paper, we have given a method for evaluation of $\mathrm{U}_q(n)$ Racah coefficients from the SDCs of Hecke algebras by using the Schur–Weyl duality relation between $\mathrm{U}_q(n)$ and $\mathrm{H}_f(q)$. The advantage of this method is that the calculation of the Racah coefficients is rank-independent. The Schur–Weyl duality relation also enable us to obtain CG coefficients of $\mathrm{U}_q(n)$ from induction coefficients (IDCs) of Hecke algebras [23]. At present our calculation is based on generic q. The situation becomes more complicated when q is a root of unity, which, however, is of importance in some applications. We will discuss the root of unity case in the near future.

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Appendix. Tables of $U_q(n)$ Racah coefficients

Table I-1a. $v=[21]$, $v_1=[1]$, $v_{12}=[11]$.

	$v_2=[1]$ $v_3=[1]$
[2]	[3]/[2]^2
[1^2]	−1/[2]^2

Table II-1a. $v=[31]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] [2]	[1] [1^2]
[3]	1/[3]	0
[21]	[4]/[3][2]	1

Table II-1b. $v=[31]$, $v_1=[1]$, $v_{12}=[1^2]$.

	[1] [2]
[3]	[4]/[3][2]
[21]	−1/[3]

Table II-1c. $v=[22]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] [2]
[21]	1

Table II-1d. $v=[22]$, $v_1=[1]$, $v_{12}=[1^2]$.

	[1] [1^2]
[21]	1

Table II-1e. $v=[211]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] [1^2]
[21]	1/[3]
[1^3]	[4]/[3][2]

Table II-1f. $v=[21^2]$, $v_1=[1]$, $v_{12}=[1^2]$.

	[1] [2]	[1] [1^2]
[21]	1	[4]/[3][2]
[1^3]	0	−1/[3]

Table II-2a. $v=[31]$, $v_1=[1]$, $v_{12}=[3]$.

	[2] [1]
[3]	1/[3]^2
[21]	[4][2]/[3]^2

Table II-2b. $v=[31]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [1]	[1^2] [1]
[3]	[4][2]/[3]^2	0
[21]	−1/[3]^2	1

Table II-2c. $v=[22]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [1]	[1^2] [1]
[21]	1	1

Table II-2d. $v=[21^2]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [1]	$[1^2]$ [1]
[21]	1	$-1/[3]^2$
$[1^3]$	0	$[4][2]/[3]^2$

Table II-2e. $v=[21^2]$, $v_1=[1]$, $v_{12}=[1^3]$.

	$[1^2]$ [1]
[21]	$[4][2]/[3]^2$
$[1^3]$	$1/[3]^2$

Table III-1a. $v=[41]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] [3]	[1] [21]
[4]	[3]/[2][4]	0
[31]	[5]/[2][4]	1

Table III-1b. $v=[41]$, $v_1=[1]$, $v_{12}=[1^2]$.

	[1] [3]
[4]	[5]/[2][4]
[31]	−[3]/[2][4]

Table III-1c. $v=[32]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] [3]	[1] [21]
[31]	1	$1/[2]^2$
$[2^2]$	0	$[3]/[2]^2$

Table III-1d. $v=[32]$, $v_1=[1]$, $v_{12}=[1^2]$.

	[1] [21]
[31]	$[3]/[2]^2$
$[2^2]$	$-1/[2]^2$

Table III-1e. $v=[31^2]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] [21]	[1] $[1^3]$
[31]	[3]/[2][4]	0
$[21^2]$	[5]/[2][4]	1

Table III-1f. $v=[31^2]$, $v_1=[1]$, $v_{12}=[1^2]$.

	[1] [3]	[1] [21]
[31]	1	[5]/[4][2]
$[21^2]$	0	−[3]/[2][4]

Table III-1g. $v=[221]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] [21]
[22]	$1/[2]^2$
[211]	$[3]/[2]^2$

Table III-1h. $v=[221]$, $v_1=[1]$, $v_{12}=[1^2]$.

	[1] [21]	[1] $[1^3]$
[22]	$[3]/[2]^2$	0
$[21^2]$	$-1/[2]^2$	1

Table III-1i. $v=[21^3]$, $v_1=[1]$, $v_{12}=[2]$.

	[1] $[1^3]$
$[21^2]$	[3]/[2][4]
$[1^4]$	[5]/[2][4]

Table III-1j. $v=[41]$, $v_1=[1]$, $v_{12}=[3]$.

	[1] [21]	[1] $[1^3]$
$[21^2]$	1	[5]/[2][4]
$[1^4]$	0	−[3]/[2][4]

Table III-2a. $v=[41]$, $v_1=[1]$, $v_{12}=[3]$.

	[2] [2]	[2] [1²]
[4]	[2]/[4][3]	0
[31]	[5][2]/[4][3]	1

Table III-2b. $v=[41]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [2]	[1²] [2]
[4]	[5][2]/[4][3]	0
[31]	−[2]/[4][3]	1

Table III-2c. $v=[32]$, $v_1=[1]$, $v_{12}=[3]$.

	[2] [2]
[31]	1/[3]
[2²]	[4]/[3][2]

Table III-2d. $v=[32]$, $v_1=[1]$, $v_{12}=[3]$.

	[2] [2]	[2] [1²]	[1²] [2]	[1²] [1²]
[31]	$\frac{[4]}{[3][2]}$	1	1	0
[22]	−1/[3]	0	0	1

Table III-2e. $v=[31^2]$, $v_1=[1]$, $v_{12}=[3]$.

	[2] [1²]
[31]	[2]/[4][3]
[21²]	[5][2]/[4][3]

Table III-2f. $v=[31^2]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [2]	[2] [1²]	[1²] [2]	[1²] [1²]
[31]	1	$\frac{[5][2]}{[4][3]}$	$-\frac{[2]}{[4][3]}$	0
[21²]	0	$-\frac{[2]}{[4][3]}$	$\frac{[5][2]}{[4][3]}$	1

Table III-2g. $v=[31^2]$, $v_1=[1]$, $v_{12}=[1^3]$.

	[1²] [2]
[31]	$\frac{[5][2]}{[4][3]}$
[21²]	[2]/[4][3]

Table III-2h. $v=[2^21]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [2]	[2] [1²]	[1²] [2]	[1²] [1²]
[2²]	1	0	0	1/[3]
[21²]	0	1	1	[4]/[3][2]

Table III-2i. $v=[2^21]$, $v_1=[1]$, $v_{12}=[1^3]$.

	[1²] [1²]
[2²]	[4]/[3][2]
[21²]	−1/[3]

Table III-2j. $v=[21^3]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [1²]	[1²] [1²]
[21²]	1	−[2]/[4][3]
[1⁴]	0	[2][5]/[4][3]

Table III-2k. $v=[21^3]$, $v_1=[1]$, $v_{12}=[1^3]$.

	[1²] [2]	[1²] [1²]
[21²]	1	[5][2]/[4][3]
[1⁴]	0	[2]/[4][3]

Table III-3a. $v=[41]$, $v_1=[1]$, $v_{12}=[31]$.

	[3] [1]	[21] [1]
[4]	$[3][5]/[4]^2$	0
[31]	$-1/[4]^2$	1

Table III-3b. $v=[41]$, $v_1=[1]$, $v_{12}=[4]$.

	[3] [1]
[4]	$1/[4]^2$
[31]	$[3][5]/[4]^2$

Table III-3c. $v=[32]$, $v_1=[1]$, $v_{12}=[31]$.

	[3] [1]	[21] [1]
[31]	1	$1/[2]^2$
$[2^2]$	0	$[3]/[2]^2$

Table III-3d. $v=[32]$, $v_1=[1]$, $v_{12}=[2^2]$.

	[21] [1]
[31]	$[3]/[2]^2$
$[2^2]$	$-1/[2]^2$

Table III-3e. $v=[31^2]$, $v_1=[1]$, $v_{12}=[31]$.

	[3] [1]	[21] [1]
[31]	1	$-1/[4]^2$
$[21^2]$	0	$[3][5]/[4]^2$

Table III-3f. $v=[31]^2$, $v_1=[1]$, $v_{12}=[21^2]$.

	[21] [1]	$[1^3]$ [1]
[31]	$[3][5]/[4]^2$	0
$[21^2]$	$1/[4]^2$	1

Table III-3g. $v=[2^21]$, $v_1=[1]$, $v_{12}=[2^2]$.

	[21] [1]
$[2^2]$	$1/[2]^2$
$[21^2]$	$[3]/[2]^2$

Table III-3h. $v=[2^21]$, $v_1=[1]$, $v_{12}=[21^2]$.

	[21] [1]	$[1^3]$ [1]
$[2^2]$	$[3]/[2]^2$	0
$[21^2]$	$-1/[2]^2$	1

Table III-3i. $v=[21^3]$, $v_1=[1]$, $v_{12}=[21^2]$.

	[21] [1]	$[1^3]$ [1]
$[21^2]$	1	$1/[4]^2$
$[1^4]$	0	$[3][5]/[4]^2$

Table III-3j. $v=[21^3]$, $v_1=[1]$, $v_{12}=[1^4]$.

	$[1^3]$ [1]
$[21^2]$	$[3][5]/[4]^2$
$[1^4]$	$-1/[4]^2$

Table III-4a. $v=[41]$, $v_1=[2]$, $v_{12}=[3]$.

	[1] [2]	[1] $[1^2]$
[3]	$[2]^2/[3]^2$	0
[21]	$[5]/[3]^2$	1

Table III-4b. $v=[41]$, $v_1=[2]$, $v_{12}=[21]$.

	[1] [2]
[3]	$[5]/[3]^2$
[21]	$-[2]^2/[3]^2$

Table III-4c. $v=[41]$, $v_1=[1^2]$, $v_{12}=[21]$.

	[1] [2]
[3]	1

Table III-4d. $v=[32]$, $v_1=[2]$, $v_{12}=[3]$.

	[1] [2]
[3]	$1/[3]^2$
[21]	$[4][2]/[3]^2$

Table III-4e. $v=[32]$, $v_1=[2]$, $v_{12}=[21]$.

	[1] [2]	[1] $[1^2]$
[3]	$[4][2]/[3]^2$	0
[21]	$-1/[3]^2$	1

Table III-4f. $v=[32]$, $v_1=[1^2]$, $v_{12}=[21]$.

	[1] [2]	[1] $[1^2]$
[21]	1	1

Table III-4g. $v=[31^2]$, $v_1=[2]$, $v_{12}=[3]$.

	[1] $[1^2]$
[21]	$[2]^2/[3]^2$
$[1^3]$	$[5]/[3]^2$

Table III-4h. $v=[31^2]$, $v_1=[2]$, $v_{12}=[21]$.

	[1] [2]	[1] $[1^2]$
[21]	1	$[5]/[3]^2$
$[1^3]$	0	$-[2]^2/[3]^2$

Table III-4i. $v=[31^2]$, $v_1=[1^2]$, $v_{12}=[21]$.

	[1] [2]	[1] $[1^2]$
[3]	$[2]^2/[3]^2$	0
[21]	$[5]/[3]^2$	1

Table III-4j. $v=[31^2]$, $v_1=[1^2]$, $v_{12}=[1^3]$.

	[1] [2]
[3]	$[5]/[3]^2$
[21]	$-[2]^2/[3]^2$

Table III-4k. $v=[2^21]$, $v_1=[2]$, $v_{12}=[21]$.

	[1] [2]	[1] $[1^2]$
[21]	1	1

Table III-4l. $v=[2^21]$, $v_1=[1^2]$, $v_{12}=[21]$.

	[1] [2]	[1] $[1^2]$
[21]	1	$-1/[3]^2$
$[1^3]$	0	$[4][2]/[3]^2$

Table III-4m. $v=[2^21]$, $v_1=[1^2]$, $v_{12}=[1^3]$.

	[1] $[1^2]$
[21]	$[4][2]/[3]^2$
$[1^3]$	$1/[3]^2$

Table III-4n. $v=[21^3]$, $v_1=[2]$, $v_{12}=[21]$.

	[1] $[1^2]$
$[1^3]$	1

Table III-4o. $v=[21^3]$, $v_1=[1^2]$, $v_{12}=[21]$.

	[1] $[1^2]$
[2]	$[2]^2/[3]^2$
$[1^3]$	$[5]/[3]^2$

Table III-4p. $v=[21^3]$, $v_1=[1^2]$, $v_{12}=[1^3]$.

	[1] [2]	[1] [1²]
[21]	1	$[5]/[3]^2$
[1³]	0	$-[2]^2/[3]^2$

Table IV-a. $v=[321]$, $v_1=[1]$, $v_{12}=[3]$.

	[2] [21]
[32]	$1/[2][4]$
[31²]	$1/[2]^2$
[2²1]	$[5]/[2][4]$

Table IV-b. $v=[321]$, $v_1=[1]$, $v_{12}=[21]$.

	[2] [3]	[2] [21] 1	[2] [21] 2	[2] [1³]	[1²] [3]	[1²] [21] 1	[1²] [21] 2	[1²] [1³]
[32]	1	$\frac{[5][3]'^2}{2[4][2][5]'}$	$-\frac{[3]^2[5]'}{2[2][4][3]'^2}$	0	0	$\frac{[5]}{2[2][4][5]'}$	$\frac{[5]'}{2[4][2]}$	0
[31²]	0	$\frac{[5]}{2[2]^2[5]'}$	$\frac{[5]'}{2[2]^2}$	1	1	$\frac{[5]'}{2[2]^2}$	$\frac{[5]}{2[2]^2[5]'}$	0
[2²1]	0	$-\frac{[5]'}{2[4][2]}$	$-\frac{[5]}{2[2][4][5]'}$	0	0	$\frac{[3]^2[5]'}{2[2][4][3]'^2}$	$-\frac{[5][3]'^2}{2[4][2][5]'}$	1

Table IV-c. $v=[321]$, $v_1=[1]$, $v_{12}=[1^3]$.

	[1²] [21]
[32]	$[5]/[2][4]$
[31²]	$-1/[2]^2$
[2²1]	$1/[2][4]$

Table IV-d. $v=[321]$, $v_1=[1]$, $v_{12}=[2^2]$.

	[21] [2]	[21] [1²]
[32]	$[3]^2/[4][2]^3$	$[5]/[4][2]^3$
[31²]	$[3]^2/[2]^4$	$[3]^2/[2]^4$
[2²1]	$-[5]/[4][2]^3$	$-[3]^2/[4][2]^3$

Table IV-e. $v=[321]$, $v_1=[1]$, $v_{12}=[31]$.

	[3] [2]	[3] [1²]	[21] [2]	[21] [1²]
[32]	1	0	$-\frac{1}{[4]^2[2]^2}$	$\frac{[3]^2}{[4]^2[2]^2}$
[31²]	0	1	$\frac{[3]^2}{[4][2]^3}$	$\frac{[5]}{[4][2]^3}$
[2²1]	0	0	$\frac{[5][3]^2}{[4]^2[2]^2}$	$\frac{[5][3]^2}{[4]^2[2]^2}$

Table IV-f. $v=[321]$, $v_1=[1]$, $v_{12}=[21^2]$.

	[21] [2]	[21] [1²]	[1³] [2]	[1³] [1²]
[32]	$\frac{[5][3]^2}{[4]^2[2]^2}$	$\frac{[5][3]^2}{[4]^2[2]^2}$	0	0
[31²]	$-\frac{[5]}{[4][2]^3}$	$-\frac{[3]^2}{[4][2]^3}$	1	0
[2²1]	$\frac{[3]^2}{[4]^2[2]^2}$	$-\frac{1}{[4]^2[2]^2}$	0	1

Table IV-g. $v=[321]$, $v_1=[2]$, $v_{12}=[21]$.

	[1] [3]	[1] [21] 1	[1] [21] 2	[1] $[1^3]$
[31]	1	$\frac{[5][3]'^2}{2[4][2][5]'}$	$\frac{[3]^2[5]'}{2[2][4][3]'^2}$	0
[22]	0	$\frac{[5]}{2[2]^2[5]'}$	$-\frac{[5]'}{2[2]^2}$	0
$[21^2]$	0	$-[5]'/2[4][2]$	$[5]/2[2][4][5]'$	1

Table IV-h. $v=[321]$, $v_1=[1^2]$, $v_{12}=[21]$.

	[1] [3]	[1] [21] 1	[1] [21] 2	[1] $[1^3]$
[31]	1	$\frac{[5]}{2[2][4][5]'}$	$-\frac{[5]'}{2[4][2]}$	0
[22]	0	$\frac{[5]'}{2[2]^2}$	$-\frac{[5]}{2[2]^2[5]'}$	0
$[21^2]$	0	$\frac{[3]^2[5]'}{2[2][4][3]'^2}$	$\frac{[5][3]'^2}{2[4][2][5]'}$	1

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